GROUPS OF GIVEN INTERMEDIATE WORD GROWTH

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Abstract. We show that there exists a finitely generated group of growth $\sim f$ for all functions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(2R) \leqslant f(R)^2 \leqslant f(\eta_+ R)$ for all R large enough and $\eta_+ \approx 2.4675$ the positive root of $X^3 - X^2 - 2X - 4$. Set $\alpha_- = \log 2/\log \eta_+ \approx 0.7674$; then all functions that grow uniformly faster than $\exp(R^{\alpha_-})$ are realizable as the growth of a group.

We also give a family of sum-contracting branched groups of growth $\sim \exp(R^\alpha)$ for a dense set of $\alpha \in [\alpha_-, 1]$.

1. Introduction

In [15], Grigorchuk discovered the first example of group with growth function strictly between polynomial and exponential. Recall that the growth function of a finitely generated group G, generated by S as a semigroup, is

$$v(R) = \#\{g \in G \mid g = s_1 \dots s_\ell,\ \ell \leqslant R, s_i \in S\};$$

it measures the volume of a ball of radius R in the Cayley graph of G. This function depends on the choice of generating set S, but only mildly: say $v \sim v'$ if there is a constant $C > 0$ such that $v(R) \leqslant v'(CR)$ and $v'(R) \leqslant v(CR)$ for all R large enough. Then the $\sim$-equivalence class of v is independent of the choice of S.

Grigorchuk's result has been considerably extended, mainly by Grigorchuk himself, who constructed uncountably many groups of intermediate growth [16]. See [1–3, 13, 23, 26], the books [20, 24], or §1.3 for a brief overview.

Nevertheless, these results only give estimates on the growth of these groups. The first actual computation (up to $\sim$) of the growth function of a group of intermediate growth appears in [6]. Here, we prove that a large class of intermediate growth functions are the growth functions of finitely generated groups. Our main result is:

Theorem A. *Let $\eta_+ \cong 2.4675$ be the positive root of $X^3 - X^2 - 2X - 4$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying*

$$f(2R) \leqslant f(R)^2 \leqslant f(\eta_+ R) \text{ for all } R \text{ large enough.} \tag{1}$$

Then there exists a finitely generated group with growth $\sim f$.

Set $\alpha_- = \log 2/\log \eta_+ \approx 0.7674$; then every submultiplicative function that grows uniformly faster (in the sense of (1)) than $\exp(R^{\alpha_-})$ is equivalent to the growth function of a group.

The growth function of a group is necessarily nondecreasing and submultiplicative ($f(R + R') \leqslant f(R)f(R')$). Note that (1) implies that f is equivalent to a monotone, submultiplicative function.

Date: 15 November 2011.

The work is supported by the ERC starting grant GA 257110 "RaWG" and the Courant Research Centre "Higher Order Structures" of the University of Göttingen.

The following examples of functions all satisfy (1); so special cases of Theorem A give:

- for every $\alpha \in [\alpha_-, 1]$, there exists a group of growth $\sim \exp(R^\alpha)$;
- there exists groups of growth $\sim \exp(R/\log R)$, of growth $\sim \exp(R/\log\log R)$, of growth $\exp(R/\log\cdots\log R)$;
- there exists a group of growth $\sim \exp(R/A(R,R)^{-1})$, for $A(m,n)$ the Ackermann function; this last growth function is faster than any subexponential primitive-recursive function;
- for every $\alpha \leqslant \beta \in [\alpha_0, 1)$, there exists a group whose growth accumulates both at $\exp(R^\alpha)$ and at $\exp(R^\beta)$; this recovers a result by Brieussel, see [10] and §1.3;
- for any functions $\exp(R^{\alpha_0}) \precsim g_- \precsim g_+ \precsim f_- \precsim f_+ \precsim \exp(R)$ satisfying (1), there exists a group whose growth function is between g_- and f_+, and takes infinitely often values in $[f_-, f_+]$ and in $[g_-, g_+]$; this essentially covers a result by Kassabov and Pak, see [21] and §1.3.

Remark 1.1. Furthermore, the group in Theorem A may be chosen to be residually finite, and of the form $W_\omega := A \wr_X G_\omega$ for a Grigorchuk group G_ω and a finite group A. Briefly, G_ω is defined by its action on the set of infinite sequences $\{1,2\}^\infty$, and X is the orbit of 2^∞; the wreath product W_ω is the semidirect product $(\mathbb{Z}/2)^X \rtimes G_\omega$ for the natural action of G_ω on X. See §2 for details.

We give illustrations in §5.1 of sequences ω and the growth of the corresponding W_ω.

If additionally $f(R)^{2+2\log\eta_+/\log R} \leqslant f(\eta_+ R)$ for all R large enough, then there exists a torsion-free, residually finite, finitely generated group with growth $\sim f$. This group may be chosen to be an extension $\widetilde{G}_\omega$ of G_ω with abelian kernel. See §2.3 for details.

If f is recursive, both of these groups may be chosen to be recursively presented, and to have solvable word problem; see §2.4 and Remark 5.4 for details.

We obtain more precise information on W_ω in some particular cases. A group G is *self-similar* if it is endowed with a homomorphism $\phi : G \to G \wr_{\{1,\dots,d\}} \mathfrak{S}_d$ from G to its permutational wreath product with the symmetric group $\mathfrak{S}_d$. The self-similar group G is *branched* if there exists a finite-index subgroup $K \leqslant G$ such that $\phi(K)$ contains K^d; see §2 for more details, or [5] for a survey of consequences of the property. A self-similar group G with proper metric $\|\cdot\|$ is *sum-contracting* if ϕ is injective and there exist constants $\lambda < 1$ (the *contraction coëfficient*) and C such that, writing $\phi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi$, we have

$$\sum_{i=1}^d \|g_i\| \leqslant \lambda \|g\| + C.$$

A weaker contraction property '$\max\|g_i\| \leqslant \lambda\|g\| + C$' is used extensively to study self-similar groups, and is at the heart of all inductive proofs. A key observation by Grigorchuk in the early 1980's was that the (somewhat counterintuitive) sum-contracting property holds for a finitely generated group (he showed that the group $G_{\overline{\mathbf{012}}}$ is sum-contracting with $d = 8$ and $\lambda = \frac{3}{4}$). Let G be sum-contracting with contraction coëfficient λ, and write $\eta = d/\lambda$. Then G has growth at most $\exp(R^{\log d/\log\eta})$. Note that there are countably many sum-contracting groups, since

such a group is determined by a choice of generators, of map ϕ, and of which of the words of length $\leqslant C/(1-\lambda)$ are trivial; see §2.4 for details.

Theorem B. *For every periodic sequence ω containing all three letters $\mathbf{0},\mathbf{1},\mathbf{2}$ and every non-trivial finite group A, the finitely generated group $W_\omega := A \wr_X G_\omega$ is branched, sum-contracting, and has growth $\sim \exp(R^\alpha)$ for some $\alpha \in (\alpha_-, 1)$.*

Furthermore, the set of growth exponents α arising in this manner is dense in $[\alpha_-, 1]$.

For these same α, there exists a torsion-free self-similar, residually finite group $\tilde{G}_\omega$ of growth $\sim n^{n^\alpha}$.

For example, $W_{\overline{\mathbf{012}}}$ has growth $\sim \exp(R^{\alpha_-})$, as was shown in [6]. The growth of W_ω for other periodic sequences is as follows:

- For $\omega = \overline{\mathbf{0}^2\mathbf{12}}$, the growth of W_ω is $\sim \exp(R^{\log 2/\log\eta})$ with $\eta \sim 2.4057$ the positive root of $X^{12} - 39X^8 + 192X^4 - 256$.
- For $\omega = \overline{\mathbf{0102}}$, the growth of W_ω is $\sim \exp(R^{\log 2/\log\eta})$ with $\eta \sim 2.4283$ the positive real root of $X^6 + 5X^4 - 8X^2 - 16$.
- For $\omega = \overline{\mathbf{0}^{t-2}\mathbf{12}}$ and $t \to \infty$, the constant η converges to 2; more precisely, η is the positive real root of $X^{3t} - (2^{t+1} + 2^{t-1} - 1)X^{2t} + (2^{2t-1} + 2^{t+2})X^t - 2^{2t}$.

  For t large, that polynomial is approximately proportional to $(X/2)^{2t} - \frac{5}{2}(X/2)^t + \frac{1}{2}$, so $\eta \approx 2\sqrt[t]{(5+\sqrt{17})/4} \approx 2(1+C/t)$ with $C = \log((5+\sqrt{17})/4)$. This gives growth $\sim \exp(R^{1-C'/(t\log 2)})$ for $C' \to C$ as $t \to \infty$.
- For $\omega = \overline{\mathbf{0}^t\mathbf{1}^t\mathbf{2}^t}$ and $t \to \infty$, the constant η also converges to 2; more precisely, η is the positive real root of

$$X^{9t} - (4\cdot 2^{3t} - 6\cdot 2^{2t} + 6\cdot 2^t - 1)X^{6t} - (2^{6t} - 6\cdot 2^{5t} + 6\cdot 2^{4t} - 4\cdot 2^{3t})X^{3t} - 2^{6t}.$$

  For t large, that polynomial is approximately a multiple of $(X/2)^{6t} - 4(X/2)^{3t} - 1$, so $\eta \approx 2\sqrt[3t]{2+\sqrt{5}} \approx 2(1+C/t)$ with $C = \log(2+\sqrt{5})$. This gives growth $\sim \exp(R^{1-C'/(t\log 2)})$ for $C' \to C$ as $t \to \infty$.
- If two finite sequences ω, ω' may be obtained one from the other by cyclic permutation of the letters and permutation of the labels $\mathbf{0},\mathbf{1},\mathbf{2}$, then the groups $W_{\overline{\omega}}$ and $W_{\overline{\omega'}}$ are isomorphic. Furthermore, if ω is the reverse of ω', then $W_{\overline{\omega}}$ and $W_{\overline{\omega'}}$ have same growth; this also happens in other cases, e.g. $\omega = \mathbf{000102211}$ and $\omega' = \mathbf{002002111}$. The corresponding groups are not isomorphic, at least if A has odd order; indeed an isomorphism would have map $G_{\overline{\omega}}$ to $G_{\overline{\omega'}}$, but these last groups are not isomorphic, see [16].

These examples were obtained with the help of Proposition 4.4, by computing the characteristic polynomials of the corresponding products of matrices (2).

1.1. **Growth of groups and wreath products.** We define a partial order on growth functions $v\colon \mathbb{R}_+ \to \mathbb{R}_+$, as follows. Say $v \precsim v'$ if, for some constants $C, D > 0$, we have $v(R) \leqslant v'(CR)$ for all $R > D$. Then $v \sim v'$ if $v \precsim v' \precsim v$.

We recall briefly the classical lemma that the equivalence class of a growth function is independent of the generating set. More generally, we shall consider *weighted word metrics* of groups, given by a weight function $\|\cdot\|\colon S \to \mathbb{R}_+^*$ for a generating set S. The *norm* of a word $w = s_1 \dots s_\ell$ is then $\|s_1\| + \dots + \|s_\ell\|$, and the norm of $g \in G$ is the infimum of norms of words representing it. The growth function is then $v(R) = \#\{g \in G \mid R \geqslant \|g\|\}$.

Consider a group A, and a group G acting on the right on a set X. Their *permutational wreath product* is $W = A \wr_X G := (\sum_X A) \rtimes G$, with the natural permutation

action of G on $\sum_X A$. If A, G are finitely generated and G acts transitively on X, then W is finitely generated. Fix a basepoint $\rho \in X$; then W is generated by $S \cup T$, with S a generating set of G and T the set of functions $X \to A$ that vanish outside ρ and take values in a fixed generating set of A at ρ.

The growth of W is intimately related to the *inverted orbit growth* of G on X: the maximum $\Delta(R)$, over all words $s_1 \dots s_\ell \in S^*$ of norm at most R, of $\#\{\rho s_1 \dots s_\ell, \rho s_2 \dots s_\ell, \dots, \rho s_\ell, \rho\}$. This was already used in [6] to determine the growth of $W_{\overline{\mathbf{012}}}$, and in [7] to prove that certain permutational wreath products have exponential growth

1.2. **Sketch of the argument.** Let us consider infinite sequences ω over the alphabet $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$ which alternates appropriately in segments of the form $(\mathbf{012})^{i_k}$ and $\mathbf{2}^{j_k}$. For any such sequence, Grigorchuk constructed in [16] a 4-generated group $G_\omega = \langle a, b, c, d\rangle$ of intermediate growth, acting on the infinite binary rooted tree $\mathcal{T}$, see §2.1. The first generator permutes the two top-level branches of $\mathcal{T}$, while the next three fix an infinite ray ρ in $\mathcal{T}$ and permute branches in the immediate neighbourhood of ρ according to ω; the ith letter of ω determines how b, c, d act on the neighbourhood of the ith point of ρ.

Let X denote the orbit of ρ under G_ω, choose a non-trivial finite group A, say $\mathbb{Z}/2\mathbb{Z}$, and define $W_\omega = A \wr_X G_\omega$, the permutational wreath product of A and G_ω; see §2.2.

We estimate the growth of W_ω by bounding from above the word growth of G_ω and by bounding from above and below the inverted orbit growth of G_ω on X, as in [6]. The difference may be summarized as follows: instead of considering a fixed group G_ω, we consider a sequence of groups G_i, and a dynamical system relating optimal choices of metrics on them. We relate the growth of the G_i to properties of the dynamical system. The example of $W_{\overline{\mathbf{012}}}$ is recovered as a 3-periodic orbit in the dynamical system.

Let G_i denote the group $G_{s^i\omega}$ for s the shift map on sequences. There are then injective homomorphisms $\phi_i \colon G_i \to G_{i+1} \wr \mathfrak{S}_2$. For each i we consider a space of metrics on G_i, see §3. These metrics are specified by setting the lengths of a, b, c, d, and are naturally parameterized by the open 2-simplex. The boundary of the simplex corresponds to degenerate metrics, in which $\|a\| = 0$, while the corners of the simplex correspond to even more degenerate metrics, in which $\|a\| = \|x\| = 0$ for some $x \in \{b, c, d\}$. The scale of the metric is set by the condition $\|ab\| + \|ac\| + \|ad\| = 1$.

We select for each i a metric $\|\cdot\|_i$ on G_i and a parameter $\eta_i \in (2, 3)$, such that the map ϕ_i is coarsely $(2/\eta_i)$-Lipschitz (i.e. satisfies $\|\phi_i(g)\|_{i+1} \leqslant 2/\eta_i \|g\|_i + C$ for a constant C) with respect to the metric on G_i and the ℓ_1 metric on $G_{i+1} \times G_{i+1}$; see Lemma 3.3. The optimal choice of metrics is studied via a projective dynamical system on the simplex. In fact, η_i depends analytically on $\|\cdot\|_i$ and the ith letter of ω. Whenever ω has long constant subwords, the metrics degenerate towards the boundary of the simplex, but do so in a controlled manner; in particular, we make sure that the metrics never degenerate to a corner of the simplex.

We arrive at the heart of the argument. The first step is to deduce, in Proposition 3.6, an upper bound on the word growth of G_i, in terms of $\eta_1 \cdots \eta_i$.

The next step, given in Proposition 3.9, deduces sharp upper and lower bounds on the inverted orbit growth of G_i on X. The first two steps together give sharp bounds on the growth of W_ω, see Corollary 4.2.

For Theorem A, the last step is to construct, out of a function f satisfying (1), a sequence ω such that $\eta_1 \cdots \eta_i$ grows appropriately in relation to f; see the precise statement in Lemma 5.3 and (9).

For Theorem B, the last step is to deduce, from expansion properties of the dynamical system on the 2-simplex, that the averages $(\eta_1 \cdots \eta_i)^{1/i}$ along periodic orbits of the dynamical system take a dense set of values in $[2, \eta_+]$.

1.3. **Prior work.** In 1983, Grigorchuk proved the existence of groups with intermediate growth function between polynomial and exponential. This answered a long-standing question asked by Milnor [25]. Since then, improved upper and lower bounds for the growth of Grigorchuk's group have been found; the best upper bound is $\exp(R^{\alpha_-}) \approx \exp(R^{0.7674})$, see [1], while the best lower bound is $\exp(R^{0.5157})$, see [2]. Note that previous articles often use the notation $\eta = 2/\eta_+$.

Grigorchuk constructed in [16, Theorem 7.2] a continuum of 2-groups G_ω of intermediate growth, for ω suitably chosen among infinite sequences over the alphabet $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. Similar constructions of p-groups for any prime p, and of torsion-free groups were given in [17]. He proved that the G_ω have nonequivalent growth functions, that their growth functions are not totally ordered, and that for every subexponentially growing function ρ there exists a group whose growth function v satisfies $v \not\precsim \rho$ and $v \prec \exp(R)$. This argument was modified in [13] to show that for every such ρ there exists a group whose growth function satisfies $\rho \prec v \prec \exp(R)$.

Grigorchuk's argument was to consider the groups G_ω for ω containing very long subsequences of the form $\mathbf{2}^{j_k}$; he showed that the growth of G_ω approaches the exponential function, at least on a subsequence. For these same sequences, we show that the growth of W_ω uniformly approaches the exponential function.

At the other extreme, if $\omega = \overline{\mathbf{012}}$, the growth of G_ω is minimal among the known superpolynomial growth functions.

Generally, good lower bounds for the growth of G_ω seem much harder to construct than good upper bounds; see [12, 14, 23].

A previous article by the authors [6] explores the growth of permutational wreath products of the form $A \wr_X G := A^X \rtimes G$, for a group A and a a group G acting on a set X. They obtained in this manner groups of intermediate growth $\sim \exp(R^\alpha)$ for a sequence of $\alpha \to 1$; those were the first examples of groups of intermediate growth for which the growth function was determined. The method of proof is an estimation of the growth of "inverted orbits" of G on X, see §3.2.

This construction was used by Brieussel [10], who gave for every $\alpha \leqslant \beta \in (\alpha_-, 1)$ a group whose growth function oscillates between $\exp(R^\alpha)$ and $\exp(R^\beta)$ in the sense that $\log\log v(n)/\log(n)$ accumulates both at α and β.

Kassabov and Pak [21] construct, for "sufficiently regular" functions $v_{G_{\overline{\mathbf{012}}}} \precsim g_- \precsim g_+ \precsim f_- \precsim f_+ \precsim \exp(R)$, a group whose growth function is between g_- and f_+, and takes infinitely often values in $[f_-, f_+]$ and in $[g_-, g_+]$. Their idea is to chose finite groups $(A_n)_{n\geqslant 0}$, and to consider an appropriate subgroup of the "permutational extension" $A \rtimes G_{\overline{\mathbf{012}}}$, for $A = \prod_{n\geqslant 0} A_n^{2^n}$ with its natural G-action.

Until recently, the analogue of Theorem A wasn't even known in the class of semigroups; though Trofimov shows in [28] that for any $f_- \gtrsim\!\!\!\!/\; R^2$ and $f_+ \lesssim\!\!\!\!/\; \exp(R)$ there exists a 2-generated semigroup with growth function infinitely often $\leqslant f_-$ and infinitely often $\geqslant f_+$. For every submultiplicative function f, a semigroup with growth between $f(R)$ and $Rf(R)$ is constructed in [8]. In particular, if $f(CR) \geqslant Rf(R)$ for some $C > 0$ and all $R \in \mathbb{N}$, then there exists a semigroup with growth

$\sim f$. In other words, all growth functions uniformly above $R^{\log R}$ are realizable as the growth of a semigroup. Warfield showed in [29] that for every $\beta \in [2, \infty)$ there exists a semigroup with growth $\sim R^\beta$; so the only known gap in growth functions of semigroups is Bergman's, between linear and quadratic [22, Theorem 2.5].

1.4. **Open problems.** We would very much like to obtain a complete description of which equivalence classes of growth functions may occur as the growth of a group. Thanks to Gromov's result [19] that groups of polynomial growth are virtually nilpotent and therefore of growth $\sim n^d$ for an integer d, we are interested in groups of intermediate growth. The groups that we construct have growth above $\exp(R^{\alpha-})$. A tantalizing open problem in the existence of groups of growth between polynomial and $\exp(\sqrt{n})$; A conjecture of Grigorchuk asserts that they do not exist. We do not even know of any groups whose growth is between polynomial and $\exp(R^{\alpha-})$.

All groups of intermediate growth known up to now satisfy a uniform lower bound on their growth, of the form $v(R)^2 \leqslant v(\eta R)$ for some η. Note that branched self-similar groups, being groups commensurable to a direct power of themselves, necessarily satisfy such an inequality. We ask whether every superpolynomial growth function of a group satisfies such an inequality, for large enough η.

2. Self-similar groups

Self-similar groups are groups endowed with a *self-similar* action on sequences, namely, an action that is determined by actions on subsequences. We adopt a more algebraic notation.

Definition 2.1. A *self-similar group* G is a group endowed with a homomorphism $\phi : G \to G \wr \mathfrak{S}_d$, for some $d \in \mathbb{N}$.

A *self-similar group sequence* is a sequence $(G_0, G_1, \dots)$ of groups, with embeddings $\phi_i : G_i \to G_{i+1} \wr \mathfrak{S}_{d_i}$ for all $i \in \mathbb{N}$.

A self-similar group gives rise to a self-similar group sequence: if $\phi : G \to G \wr \mathfrak{S}_d$, then set $G_i = G$, $d_i = d$ and $\phi_i = \phi$ for all $i \in \mathbb{N}$.

Let (G_i) be a self-similar group sequence, the d_i and ϕ_i being implicit in the notation. Define

$$T_0 = \bigsqcup_{i \in \mathbb{N}} \{1, \dots, d_0\} \times \cdots \times \{1, \dots, d_{i-1}\}.$$

This is the vertex set of a rooted tree (with root the empty product $\varnothing$), if one puts an edge between $x_0 \dots x_{i-1}$ and $x_0 \dots x_{i-1} x_i$ for all $x_j \in \{1, \dots, d_j\}$. We denote also this tree by T_0.

The group G_0 acts by isometries on T_0 as follows. Given $g \in G$ with $\phi_0(g) = (f, \pi)$ and $x_0 \dots x_{i-1} \in T_0$, set

$$(x_0 \dots x_{i-1})^g = \begin{cases} \varnothing & \text{if } i = 0, \\ (x_0^\pi)\,(x_1 \dots x_{i-1})^{f(x_0)} & \text{inductively if } i > 0. \end{cases}$$

The boundary of T_0 is naturally identified with infinite sequences

$$\partial T_0 = \prod_{i \in \mathbb{N}} \{1, \dots, d_i\},$$

and the action of G on T_0 extends to a continuous action on ∂T_0.

Definition 2.2. The self-similar group sequence (G_i) is *branched* if there exist for all $i \in \mathbb{N}$ a finite-index subgroup $K_i \leqslant G_i$ such that $K_{i+1}^{d_i} \leqslant \phi_i(K_i)$ for all $i \in \mathbb{N}$.

For more information on branch groups, and the algebraic consequences that follow from that property, we refer to [5].

2.1. **The Grigorchuk groups G_ω.** Our main examples were constructed by Grigorchuk in [16]. Let $\{\mathbf{0},\mathbf{1},\mathbf{2}\}$ denote the three non-trivial homomorphisms from the four-group $\{1,b,c,d\}$ to the cyclic group $\{1,a\}$, ordered for definiteness by the condition that, in that order, they vanish on b,c,d respectively. Write $\Omega=\{\mathbf{0},\mathbf{1},\mathbf{2}\}^{\mathbb{N}}$ the space of infinite sequences over $\{\mathbf{0},\mathbf{1},\mathbf{2}\}$, endowed with the shift map $s\colon\Omega\to\Omega$ given by $s(\omega_0\omega_1\dots)=\omega_1\omega_2\dots$.

Fix a sequence $\omega\in\Omega$, and define a self-similar group sequence as follows. Take $d=2$, and write $\mathfrak{S}_d=\{1,a\}$. Each G_i is generated by $\{a,b,c,d\}$. Set

$$\phi_i(a)=\langle\!\langle 1,1\rangle\!\rangle a,\quad \phi_i(b)=\langle\!\langle \omega_i(b),b\rangle\!\rangle,\quad \phi_i(c)=\langle\!\langle \omega_i(c),c\rangle\!\rangle,\quad \phi_i(d)=\langle\!\langle \omega_i(d),d\rangle\!\rangle.$$

The assertion that the ϕ_i are homomorphisms, and that the G_i act faithfully on the binary rooted tree, define the groups G_i uniquely.

Define the *Grigorchuk group* G_ω as the group G_0 constructed above. We then have $G_i=G_{s^i\omega}$ for all $i\geqslant 0$, so there are homomorphisms $\phi_\omega: G_\omega\to G_{s\omega}\wr\mathfrak{S}_2$, and G_0 is the first entry of a self-similar group sequence.

If ω is not ultimately constant, it is known that each of the G_ω have intermediate growth, with growth bounded from below by $\exp(R^{1/2})$ and from above by $\exp(R)$; and no smaller function may serve as upper bound [16, Theorem 7.1].

(This may be also seen concretely as follows: consider the "universal" group $\widehat{G}$, defined as the diagonal subgroup generated by $\hat a=(a,a,\dots),\hat b=(b,b,\cdots),\hat c=(c,c,\dots),\hat d=(d,d,\dots)$ in $\prod_{\omega\in\Omega}G_\omega$. Then this group has exponential growth — indeed, $\{\hat a\hat b,\hat a\hat c,\hat a\hat d\}$ freely generates a free semigroup.)

Proposition 2.3. *The group G_ω is the first term of a self-similar group sequence;*

(1) *if ω contains only a finite number of one of the three symbols, then G_ω contains an element of infinite order;*
(2) *if the sequence ω contains infinitely many of each of the three symbols, then G_ω is an infinite, branched, torsion group;*
(3) *if ω is periodic and contains each of the three symbols $\mathbf{0},\mathbf{1},\mathbf{2}$, then G_ω is self-similar.*

Proof. It was already proven in [16] that G_ω is torsion if and only if ω contains all three symbols.

To show that G_ω are branched, consider the groups $G_i=G_{s^i\omega}$ and homomorphisms $\phi_i: G_i\to G_{i+1}\wr\mathfrak{S}_2$. For each $i\in\mathbb{N}$, let $x_i\in\{b,c,d\}$ be such that $\omega_i(x_i)=1$, and set $K_i=\langle[x_i,a]\rangle^{G_i}$. Choose also $y_i\in\{b,c,d\}\backslash\{x_i\}$. Then $1\times K_{i+1}$ is normally generated by $\langle\!\langle 1,[x_{i+1},a]\rangle\!\rangle$, and

$$\langle\!\langle 1,[x_{i+1},a]\rangle\!\rangle=\phi_i([x_i,y_i^a])=\phi_i([x_i,a][x_i,a]^{ay_i})\in\phi_i(K_i);$$

the same holds for $\langle\!\langle [x_{i+1},a],1\rangle\!\rangle$.

Consider first $G_i/\langle x_i\rangle^{G_i}$. This group is generated by two involutions a and y_i, so is a finite dihedral group, because G_i is torsion. It follows that $\langle x_i\rangle^{G_i}$ has finite index in G_i. Then K_i has index 2 in $\langle x_i\rangle^{G_i}$, so also has finite index.

Finally, if ω is periodic, say of period n, then $G_0=G_n$ and the composition of the maps ϕ_i gives a homomorphism $\phi: G_0\to G_0\wr\mathfrak{S}_{2^n}$, showing that G_ω is self-similar. $\square$

2.2. **The Groups W_ω.** Fix a finitely generated group A, an infinite sequence $\rho \in \{1,2\}^\infty$ such as $\rho = 2^\infty$, and its orbit $X := \rho \cdot G_\omega$. Consider $W_\omega := A \wr_X G_\omega$.

Proposition 2.4. *The group W_ω is finitely generated, and is the first term of a branched self-similar group sequence.*

If ω and ρ are periodic, then W_ω is also self-similar.

If ω and ρ are recursive and A is recursively presented, then so is W_ω. If moreover A has a solvable word problem, then so does W_ω.

If A is torsion and ω contains infinitely many of all three symbols, then W_ω is torsion.

Proof. Clearly W_ω is generated by $\{a,b,c,d\} \cup S$ for a finite generating set S of A; we imbed A in W_ω as those functions $X \to A$ supported on $\{\rho\}$.

Let (G_i) be the self-similar group sequence with $G_0 = G_\omega$. Let ρ_i be the subsequence of ρ starting at position i. Define $W_i = A \wr_{\rho_i G_i} G_i$, with imbeddings $\psi_i : W_i \to W_{i+1} \wr \mathfrak{S}_2$ given as follows. Consider $w = (u,g) \in W_i$, with $u\colon \rho_i G \to A$ and $g \in G_i$. Write $\phi_i(g) = (f,\pi)$ with $f\colon \{1,2\} \to G_{i+1}$ and $\pi \in \mathfrak{S}_2$. For $k \in \{1,2\}$, set $u_k(x) := u(kx)\colon \rho_{i+1}G_{i+1} \to A$. Then

$$\psi_i(u,g) = (k \mapsto (u_k, f(k)), \pi).$$

Furthermore, let K_i be the finite-index subgroup of G_i for which (G_i) is branched. Then $L_i := A \wr K_i$ has finite index in W_i, and shows that (W_i) is branched.

If ω is periodic, say of period n, then $G_0 = G_n$ and we get, composing the maps $\phi_0, \dots, \phi_n$, an imbedding $\phi : G_0 \to G_0 \wr \mathfrak{S}_{2^n}$, showing that G_ω is self-similar. The same holds for W_ω.

For a presentation of W_ω, see [4], [11] and [6]. Let A be generated by a finite set S; then W_ω is generated by $S \sqcup \{a,b,c,d\}$. The relations of W_ω are the following: those of A; those of G_ω; commutation relations $[s,(s')^g]$ for all $s,s' \in S$ and $g \in G_\omega$ not fixing ρ; and commutation relations $[s,g]$ for all $s \in S$ and $g \in G_\omega$ fixing ρ.

If ω is recursive, then the action of G_ω on X is computable, so the sets $\{w \in \{a,b,c,d\}^* \mid \rho w = w\}$ and $\{w \in \{a,b,c,d\}^* \mid \rho w \neq w\}$ are recursive; so W_ω is recursively presented.

If moreover the word problem is solvable in A, then it is also solvable in W_ω. Indeed G_ω is contracting, so the word problem is solvable in G_ω; see §2.4. A word in W_ω is trivial if and only if its image in G_ω is trivial and its values at all $x \in X$ are trivial in A.

If ω contains infinitely many of all three symbols, then G_ω is torsion; if moreover A is torsion, then W_ω, being an extension of two torsion groups, is also torsion. $\square$

2.3. **The torsion-free groups $\tilde G_\omega$.** Grigorchuk constructed in [17, §5] a torsion-free group $\tilde G$ of intermediate growth; it was later noted in [18] that $\tilde G$ acts continuously on the interval $[0,1]$. We shall consider variants $\tilde G_\omega$ of his construction, which we now recall.

The group $\tilde G_\omega$ is a self-similar group in the sense of Definition 2.1, but the map $\phi : \tilde G_\omega \to \tilde G_{s\omega} \wr \mathfrak{S}_2$ is not injective. We construct the groups $\tilde G_\omega$ via a slightly different definition. Each of the groups $\tilde G_\omega$ acts faithfully on $\mathbb{Z}^\infty$, the rooted tree of infinite arity. Each $\tilde G_\omega$ is generated by $\{a,b,c,d\}$. Each $\omega \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}$ is now understood as a homomorphism $\langle b,c,d\rangle \cong \mathbb{Z}^3 \to \mathbb{Z} \cong \langle a\rangle$, via $\mathbf{0}(b) = 1$, $\mathbf{0}(c) = \mathbf{0}(d) = a$ etc. under cyclic permutation. There is a homomorphism $\tilde\phi_\omega\colon \tilde G_\omega \to \prod_{\mathbb{Z}} \tilde G_\omega \rtimes \mathbb{Z}$, given

by

$$\tilde\phi_\omega(a) = \langle\!\langle \dots, 1, 1, \dots \rangle\!\rangle(n \mapsto n+1),$$
$$\tilde\phi_\omega(x) = \langle\!\langle \dots, \omega_0(x), x, \dots \rangle\!\rangle \text{ for all } x \in \langle b, c, d\rangle,$$

where the ... indicate that the sequence repeats 2-periodically. In particular, the map $a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d$ extends to an epimorphism $\xi_\omega \colon \tilde G_\omega \to G_\omega$.

We proved in [6, Lemma 5.7], that $\ker \xi_{\overline{\mathbf{012}}}$ is abelian, isomorphic to $\mathbb{Z} \times \sum_X \mathbb{Z}^3$; the argument extends with no changes to $\ker \xi_\omega$ for arbitrary ω. We proved in Proposition 5.8 that $\tilde G_{\overline{\mathbf{012}}}$ has growth function equivalent to $\mathbb{Z}^3 \wr G_{\overline{\mathbf{012}}}$; again, the argument extends without changes to all $\tilde G_\omega$.

The following result essentially appears in [18]; there, Grigorchuk and Machì show that $\tilde G_{\overline{\mathbf{012}}}$ is residually virtually nilpotent and residually solvable.

Lemma 2.5. *The group $\tilde G_\omega$ is residually 2-finite.*

Proof. For all $n \in \mathbb{N}$, let π_n denote the homomorphism $G_\omega \to \mathfrak{S}_{2^n}$ defined by restricting the action of G_ω to sequences in $\{1,2\}^n$, and let N_n be the normal closure in $\tilde G_\omega$ of $\langle a^{2^n}, b^{2^n}, c^{2^n}, d^{2^n}\rangle$.

Consider then the normal subgroups $P_n = N_n \ker(\pi_n \circ \xi_\omega)$ of $\tilde G_\omega$. On the one hand, $\tilde G_\omega/P_n$ is a finite 2-group: it is a torsion abelian 2-extension of the finite 2-group $\pi_n(G_\omega)$. On the other hand, $P_\infty := \bigcap_{n\in\mathbb{N}} P_n$ is trivial: P_∞ is contained in $\ker \xi_\omega \cong \mathbb{Z} \times \sum_X \mathbb{Z}^3$, and is 2-divisible in $\tilde G_\omega$. □

2.4. **Presentations.** Assume that ω is a recursive sequence; that is, there is an algorithm that, given $n \in \mathbb{N}$ as input, computes the nth letter of ω. We then claim that G_ω and $\tilde G_\omega$ have a solvable word problem; that is, there are algorithms that, given a word $w \in \{a,b,c,d\}^*$ as input, compute whether $w = 1$ in G_ω and whether $w = 1$ in $\tilde G_\omega$.

Indeed, the following algorithm solves the word problem in G_ω. Given $w \in \{a,b,c,d\}^*$, first perform all elementary cancellations $a^2 \to 1, b^2 \to 1, \dots, bc \to d, \dots$. If w became the empty word, then $w = 1$ in G_ω. Otherwise, if the exponent sum of a in w is non-zero, then $w \neq 1$ in G_ω. Otherwise, compute ω_0, and using it compute $\phi_0(w) = \langle\!\langle w_1, w_2\rangle\!\rangle$. Noting that $s\omega$ is again a recursive sequence, apply the algorithm recursively to w_1 and w_2, to determine whether both are trivial in $G_{s\omega}$.

The algorithm terminates because G_ω is contracting.

A small modification of that algorithm solves the word problem in $\tilde G_\omega$. Given $w \in \{a^{\pm1}, b^{\pm1}, c^{\pm1}, d^{\pm1}\}^*$, first perform all elementary substitutions $cb \to bc, dc \to cd, db \to bd$ and free cancellations. If w became the empty word, then $w = 1$ in $\tilde G_\omega$. Otherwise, if the exponent sum of a in w is non-zero, or if $w \in \langle b,c,d\rangle$, then $w \neq 1$ in $\tilde G_\omega$. Otherwise, compute ω_0, and using it compute $\tilde\phi_0(w) = \langle\!\langle w_1, w_2\rangle\!\rangle$. Noting that $s\omega$ is again a recursive sequence, apply the algorithm recursively to w_1 and w_2, to determine whether both are trivial in $\tilde G_{s\omega}$.

Corollary 2.6. *If ω is recursive, then G_ω and $\tilde G_\omega$ have solvable word problem, and are recursively presented.*

Proof. The solution of the word problem is given by the algorithms above. The groups are then recursively presented, by taking as relators, e.g., all words that are trivial in them. □

More efficient recursive presentations are described in [27].

Note that (unless ω is eventually constant) the group G_ω is not finitely presented. In fact, no finitely presented example of branched group with injective ϕ is known.

On the other hand, contracting groups may be described by a finite amount of data, and this immediately implies that there are countably many such groups. Let G be a contracting group, with self-similarity map $\phi : G \hookrightarrow G \wr \mathfrak{S}_d$. Fix the notation $\phi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi_g$. The contracting property implies that there are constants $\lambda < 1$ and C such that $\|g_i\| \leqslant \lambda \|g\| + C$.

Fix a generating set S for G. The map ϕ is determined by its values on generators, namely by $\#S$ permutations in $\mathfrak{S}_d$ and $d\#S$ words in the free group on S. The corresponding map $\phi : F_S \to F_S \wr \mathfrak{S}_d$ is not contracting, but there exists a finitely presented quotient $\widehat{G}$ of F_S, with $F_S \twoheadrightarrow \widehat{G} \twoheadrightarrow G$, for which the induced map $\phi : \widehat{G} \to \widehat{G} \wr \mathfrak{S}_d$ is contracting (but not injective).

The contraction property implies the following: for any $g \in \widehat{G}$, sufficiently many applications of ϕ and projection to a coördinate g_i result in elements of norm at most $B := (C+1)/(1-\lambda)$. There are finitely many elements in the ball of radius B in $\widehat{G}$; let us fix which of these elements project to the identity in G.

We claim that these data — the map $\phi : S \to F_S \wr \mathfrak{S}_d$, the finitely presented group $\widehat{G}$, and the set W of words of length at most B in G projecting to the identity in G — determine G completely. To see that, it is sufficient to exhibit an algorithm solving the word problem in G.

Given $g \in G$, written as a word in S, do the following. If $\|g\| \leqslant B$, then $g = 1$ if and only if it belongs to W. Otherwise, compute $\phi(g) = \langle\!\langle g_1, \dots, g_d \rangle\!\rangle \pi_g$. If $\pi_g \neq 1$, then $g \neq 1$. Otherwise, apply recursively the algorithm to determine whether $g_1, \dots, g_d$ are all trivial in G. This is well-founded because (by the contraction property) we have $\|g_i\| \leqslant \|g\| - 1$.

3. Metrics on the Grigorchuk groups G_ω

Let G_ω be a Grigorchuk group as above; it is the first entry of a self-similar group sequence with $G_i = G_{s^i\omega}$. We put a word metric on each G_i, i.e. a metric $\|\cdot\|_i$ defined by assigning a norm to each generator and extending it naturally to group elements. This metric is determined by a point V_i in the open simplex

$$\Delta = \big\{(\beta,\gamma,\delta) \mid \max\{\beta,\gamma,\delta\} < \frac{1}{2}, \beta+\gamma+\delta = 1\big\}$$

with vertices $(\frac12,\frac12,0)$, $(\frac12,0,\frac12)$, $(0,\frac12,\frac12)$. It assigns norm $\alpha := 1 - 2\max\{\beta,\gamma,\delta\}$ to a, and norms $\beta-\alpha, \gamma-\alpha, \delta-\alpha$ to b, c, d respectively.

In particular, the norm of the longest of b, c, d equals the sum of the norms of the two shortest.

In order to appropriately construct metrics on the G_i, we consider the space $\Delta \times \Omega$, and define three 3×3 matrices M_x, for $x \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}$, by

$$M_{\mathbf{0}} = \begin{pmatrix}1&1&1\\0&2&0\\0&0&2\end{pmatrix}, M_{\mathbf{1}} = \begin{pmatrix}2&0&0\\1&1&1\\0&0&2\end{pmatrix}, M_{\mathbf{2}} = \begin{pmatrix}2&0&0\\0&2&0\\1&1&1\end{pmatrix}. \tag{2}$$

Define a continuous function $\eta\colon \Delta\times\{\mathbf{0},\mathbf{1},\mathbf{2}\} \to (2,3]$ by the condition $\eta(p,x)^{-1}M_x(p) \in \Delta$, and projective maps $\overline{M}_x\colon \Delta \to \Delta$ by the $\overline{M}_x(p,x) = \eta(p,x)^{-1}M_x(p)$. Define

next a shift map $s\colon \Delta\times\Omega\to\Delta\times\Omega$ by

$$s(p,\omega_0\omega_1\dots)=(\overline{M}_{\omega_0}(p),\omega_1\omega_2\dots).$$

Define finally a continuous function $\mu\colon \Delta\to(0,1]$ by

$$\mu(\beta,\gamma,\delta)=\min\{\beta,\gamma,\delta\}.$$

The function $\mu(p)$ is equivalent to the minimal euclidean distance from $p\in\Delta$ to a vertex of Δ.

Note that the $\overline{M}_x$ have disjoint images, and that the union of their images is dense in Δ. More precisely, each $\overline{M}_x$ maps Δ to the simplex spanned by two vertices and the barycentre of Δ. The shift map $s\colon \Delta\times\Omega\to\Delta\times\Omega$ is injective with dense image. Define $\Delta'\subset\Delta$ by

$$\Omega^+=\bigcap_{k\geqslant 0}s^k(\Delta\times\Omega)=\Delta'\times\Omega.$$

Then Δ' is dense in Δ, being the complement of countably many line segments, and $s\colon\Omega^+\to\Omega^+$ is bijective. The metrics on the G_i defined by $p\in\Delta'$ are non-degenerate.

Endow Δ with the Hilbert metric $d_\Delta(V_1,V_2)=\log(V_1,V_2;V_-,V_+)$, computed using the cross-ratio of the points V_1,V_2 and the intersections V_-,V_+ of the line containing V_1,V_2 with the boundary of Δ. Endow Ω with the Bernoulli metric $d(\omega,\omega')=\sum_{i\colon\omega_i\neq\omega'_i}2^{-i}$.

Lemma 3.1 (Essentially [9]; we have not been able to locate a proof of the following extension)**.** *Let K be a convex subset of a real vector space, and let $A:K\to K$ be a projective map. Then A is 1-Lipschitz for the Hilbert metric.*

If furthermore $A(K)$ contains no lines from K (that is, $A(K)\cap\ell\neq K\cap\ell$ for every line ℓ intersecting K), then A is strictly contracting.

Proof. Since A is projective, it preserves the cross-ratio on lines, so we have $d_{A(K)}(A(V_1),A(V_2))=d_K(V_1,V_2)$ for all $V_1,V_2\in K$. Furthermore, on the line ℓ through V_1,V_2, the intersection points $\ell\cap\partial A(K)$ are not further from V_1,V_2 than $\{V_+,V_-\}=\ell\cap\partial K$; the Hilbert metric decreases as $V_\pm$ are moved apart from V_1,V_2, and this gives strict contraction under the condition $A(K)\cap\ell\neq K\cap\ell$. □

Lemma 3.2. *The homeomorphism $s\colon\Omega^+\to\Omega^+$ is 'hyperbolic': it decomposes as the product of an expanding and contracting map.*

Proof. The expanding direction is Ω; indeed the Bernoulli metric is doubled by the shift map. The contracting direction is Δ; indeed the maps $\overline{M}_x$ are projective, hence contracting by Lemma 3.1. Note that the contraction is strict, but not uniform. □

Let $V_0\in\Delta'$ be a given metric on G_ω; this defines a point $(V_0,\omega)\in\Omega^+$. We then let V_i be the first coördinate of $s^i(V_0,\omega)$, and have defined a metric $\|\cdot\|_i$ on each G_i. We also set $\eta_i=\eta(V_i,\omega_i)$ and $\mu_i=\mu(V_i)$, to lighten notation.

(Note that the choice of $V_0\in\Delta'$ will ultimately not be very important, because the maps $\overline{M}_x$ are contracting. Indeed, for two choices V_0,V'_0 we have $V_i\to V'_i$, so $\mu_i\to\mu'_i$ and $\eta_i\to\eta'_i$ as $i\to\infty$.)

Lemma 3.3. *For each $i\geqslant 0$ and each $g\in G_i$, with $\phi_i(g)=\langle\!\langle g_1,g_2\rangle\!\rangle\pi$, we have*

$$\|g_1\|_{i+1}+\|g_2\|_{i+1}\leqslant\frac{2}{\eta_i}(\|g\|_i+\|a\|_i).$$

Proof. We consider first $\omega_i = \mathbf{0}$; the general case will follow. Consider $V_i = (\beta_i, \gamma_i, \delta_i) \in \Delta$, and $V_{i+1} = \overline{M}_{\mathbf{0}}(V_i) = (\beta_{i+1}, \gamma_{i+1}, \delta_{i+1})$. We compute $\eta_i = \eta(V_i, \mathbf{0}) = 3 - 2\beta_i$, and $V_{i+1} = (1/\eta_i, 2\gamma_i/\eta_i, 2\delta_i/\eta_i)$, so

$$\|b\|_{i+1} = \beta_{i+1} - \|a\|_{i+1} = \beta_{i+1} - 1 + 2\beta_{i+1} = 3/\eta_i - 1 = 2\beta_i/\eta_i.$$

The same result holds, cyclically permuting β, γ, δ, for all $\overline{M}_x$. We therefore have

$$\begin{aligned} \|b\|_{i+1} + \|\omega_i(b)\|_{i+1} &= 2\beta_i/\eta_i = 2/\eta_i(\|b\|_i + \|a\|_i), \\ \|c\|_{i+1} + \|\omega_i(c)\|_{i+1} &= 2\gamma_i/\eta_i = 2/\eta_i(\|c\|_i + \|a\|_i), \\ \|d\|_{i+1} + \|\omega_i(d)\|_{i+1} &= 2\delta_i/\eta_i = 2/\eta_i(\|d\|_i + \|a\|_i). \end{aligned} \tag{3}$$

On the other hand, g may by written as an alternating word in a and b, c, d, because of the relations $b^2 = c^2 = d^2 = bcd = 1$. Group each b, c, d-letter with an a, and sum the corresponding inequalities; there may be an a left over. This gives the desired inequality. □

Even though the metrics we consider on G_ω are not word metrics, the volume of small balls are well understood. We include the following result for future reference, though it will not be used in this article:

Lemma 3.4. *There are two absolute constants $K_1, K_2 > 0$ such that, for every group G_ω with metric given by $V_0 \in \Delta$, we have*

$$\frac{K_1}{\mu_0} \leqslant v(1/2) \leqslant \frac{K_2}{\mu_0}.$$

Proof. Assume without loss of generality $\beta \leqslant \gamma \leqslant \delta$. A word of norm $\leqslant 1$ and length $\geqslant 4$ may contain only as and bs, because otherwise it would have norm at least $\beta + \gamma > 1$. It must therefore be of the form $b^i(ab)^j a^k$ for $i, k \in \{0, 1\}$ and $0 \leqslant j \leqslant \beta^{-1}$. □

Lemma 3.5. *For every $A > 0$ there exists a constant K, such that, for all $V_0 \in \Delta'$, we have $v(A\mu_0) \leqslant K$.*

Proof. The minimal norm in $(G_\omega, \|\cdot\|_{V_0})$ of a non-trivial product of two generators is μ_0; we can therefore take $K = 4^{2A}$. □

3.1. Volume growth of G_ω, upper bound.

Let $v_i(R)$ denote the volume growth of G_i for the metric $\|\cdot\|_i$. The following proposition, and its variants in §3.2, are key to our computation of growth functions.

Proposition 3.6. *There is an absolute constant B such that for all $k \in \mathbb{N}$, we have*

$$v_0(\eta_0\eta_1 \cdots \eta_{k-1}\mu_k) \leqslant B^{2^k}.$$

Before embarking in the proof, we give a few preliminary definitions and results. For a non-negative function $f(R)$, its *concave majorand* is the smallest concave function f^+ greater than f; the graph of f^+ bounds the convex hull of the graph of f. It may be defined by

$$f^+(R) = \sup_{p<R<q} \left(\frac{q-R}{q-p} f(p) + \frac{R-p}{q-p} f(q) \right). \tag{4}$$

The following lemma is not used in the argument, but only included for illustration. It implies that the growth functions considered in Theorem A are equivalent to log-concave functions.

Lemma 3.7. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying* (8)*. Then g is equivalent to a concave function.*

Proof. First, the inequality $g(2R) \leqslant g(R)$ alone implies that g is equivalent to a subadditive function, namely a function satisfying $g(R+S) \leqslant g(R)+g(S)$. We therefore assume, without loss of generality, that g is subadditive. Let g^* be the concave majorand of g, as in (4). We will show $g^* \leqslant 2g$; this with the second inequality in (8) yields the claim.

Without loss of generality, assume also that g is increasing. Consider $a < x < b$ such that $g^*(x) = (x-a)/(b-a)g(b) + (b-x)/(b-a)g(a)$. Write $b = Nx + c$ with $0 \leqslant c < x$. Then $(b-a)g^*(x) = (x-a)g(Nx+c) + ((N-1)x+c)g(a) \leqslant (x-a)(Ng(x)+g(c)) + ((N-1)x+c)g(a) \leqslant (x-a)(N+1)g(x) + ((N-1)x+c)g(x) = (2Nx+c-(N+1)a)g(x) \leqslant 2(b-a)g(x)$, so $g^*(x) \leqslant 2g(x)$. □

Lemma 3.8. *For all $k \in \mathbb{N}$ we have $\eta_k\mu_{k+1} \geqslant \mu_k + \|a\|_k$.*

Proof. We write $V_k = (\beta,\gamma,\delta)$, and assume without loss of generality $\omega_k = \mathbf{0}$. Then the inequality

$$\eta_k\mu_{k+1} = 2\min\{\gamma,\delta\} \geqslant \min\{\beta,\gamma,\delta\} + (1-\max\{\beta,\gamma,\delta\}) = \mu_k + \|a\|_k$$

is checked by considering the six cases $\beta < \gamma < \delta$ etc. in turn. □

Proof of Proposition 3.6. By Lemma 3.3 we have

$$\begin{aligned} v_i(R) &\leqslant \int_{p_0+p_1 = 2/\eta_i(R+\|a\|_i)} 2v_{i+1}(p_0)v_{i+1}(p_1) \\ &\leqslant 2\big(1+\frac{2}{\eta_i}(R+\|a\|_i)\big) \max_{p_0+p_1=2/\eta_i(R+\|a\|_i)} v_{i+1}(p_0)v_{i+1}(p_1). \end{aligned} \tag{5}$$

Set $\alpha_i(R) := (\log v_i)^+(R)$. Below, we abbreviate in '$\cdots$' the development of $\log v_i(q)$, which is similar to that of $\log v_i(p)$:

$$\begin{aligned} \alpha_i(R) &= \sup_{p<R<q} \left\{ \frac{q-R}{q-p}\log v_i(p) + \frac{R-p}{q-p}\log v_i(q) \right\} \\ &\leqslant \sup_{p<R<q} \left\{ \frac{q-R}{q-p} \log \int_{p_0+p_1=\frac{2}{\eta_i}(p+\|a\|_i)} 2v_{i+1}(p_0)v_{i+1}(p_1) + \frac{R-p}{q-p}\cdots \right\} \text{ by (5)} \\ &\leqslant \log 2 + \sup_{p<R<q} \left\{ \frac{q-R}{q-p} \log \int_{p_0+p_1=\frac{2}{\eta_i}(p+\|a\|_i)} \exp\big(\alpha_{i+1}(p_0)+\alpha_{i+1}(p_1)\big) + \frac{R-p}{q-p}\cdots \right\} \\ &\leqslant \log 2 + \sup_{p<R<q} \left\{ \frac{q-R}{q-p} \log \int_{p_0+p_1=\frac{2}{\eta_i}(p+\|a\|_i)} \exp\left(2\alpha_{i+1}\big(\frac{p+\|a\|_i}{\eta_i}\big)\right) \right. \\ &\qquad\qquad \left. + \frac{R-p}{q-p}\cdots \right\} \text{ because } \alpha_{i+1} \text{ is concave} \\ &\leqslant \log 2 + \sup_{p<R<q} \left\{ \frac{q-R}{q-p} \left[\log\left(1+\frac{2(p+\|a\|_i)}{\eta_i}\right) + 2\alpha_{i+1}\big(\frac{p+\|a\|_i}{\eta_i}\big) \right] + \frac{R-p}{q-p}\cdots \right\} \\ &\leqslant \log 2 + \log\left(1+\frac{2(R+\|a\|_i)}{\eta_i}\right) + 2\alpha_{i+1}\big(\frac{R+\|a\|_i}{\eta_i}\big) \end{aligned}$$

because α_{i+1} and $\log(1+\cdots)$ are concave.

Set then $\beta_i(R) = \alpha_i(R+\mu_i) + \log(1+R+\mu_i) + 2\log 2 + 2\log 3$. We will in fact prove the stronger inequality $\beta_0(\eta_0\eta_1\cdots\eta_{k-1}\mu_k) \leqslant 2^k B'$ for an absolute constant B'. We have

$$\begin{aligned}\beta_i(R) &= \alpha_i(R+\mu_i) + \log(1+R+\mu_i) + 2\log 2 + 2\log 3\\ &\leqslant 3\log 2 + 2\log 3 + \log(1+R+\mu_i) + \log\Big(1+\frac{2(R+\|a\|_i)}{\eta_i}\Big) + 2\alpha_{i+1}\Big(\frac{R+\mu_i+\|a\|_i}{\eta_i}\Big)\\ &\leqslant 2\left(2\log 2 + \log 3 + \log\eta_i + \log\big(1+\frac{R}{\eta_i}+\mu_{i+1}\big) + \alpha_{i+1}\big(\frac{R}{\eta_i}+\mu_{i+1}\big)\right) \text{ by Lemma 3.8}\\ &\leqslant 2\beta_{i+1}(R/\eta_i).\end{aligned}$$

We therefore have

$$\alpha_0(\eta_0\cdots\eta_k\mu_k) \leqslant \beta_0(\eta_0\cdots\eta_{k-1}\mu_k) \leqslant 2^k\beta_k(\mu_k) \leqslant \alpha_k(2\mu_k) + \log(1+2\mu_k) + \log 36,$$

and $\alpha_k(2\mu_k)$ is bounded by Lemma 3.5. □

3.2. **Inverted orbit growth of G_ω.** Recall that, for a group G acting on a set X on the right and a basepoint $* \in X$, and for a word $w = g_1 \dots g_R \in G^*$, we denote by $\Delta(w)$ the cardinality of the *inverted orbit* $\{*g_1\dots g_R, *g_2\dots g_R, \dots, *g_R, *\}$ of $*$ under w. We denote by $\Delta(R)$ the maximal inverted orbit growth of words $w \in S^R$. Following the notation used before, we denote by $\Delta_i(R)$ the maximal size of an inverted orbit under a word in G_i of norm at most R in the metric $\|\cdot\|_i$.

Proposition 3.9. *There is an absolute constant C such that for all $k \in \mathbb{N}$ we have*

$$2^k \leqslant \Delta(\eta_0\cdots\eta_{k-1}\mu_k) \leqslant C2^k.$$

Proof of the upper bound. The calculation in this section follows closely that of the previous §. Let again Δ_i^+ denote the concave majorand of Δ_i. Lemma 3.3 gives

$$\Delta_i^+(\eta_i R) \leqslant 2\Delta_{i+1}(R + \|a\|_i/\eta_i).$$

The same argument as above applies, and we do not repeat it. □

On the other hand, we also obtain a lower bound on the inverted orbit growth, using a substitution. For that purpose, define self-substitutions ζ_x of $\{ab, ac, ad\}$, for $x \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}$, by

$$\begin{array}{llll}\zeta_{\mathbf{0}}: & ab \mapsto adabac, & ac \to acac, & ad \to adad,\\ \zeta_{\mathbf{1}}: & ab \mapsto abab, & ac \to abacad, & ad \to adad,\\ \zeta_{\mathbf{2}}: & ab \mapsto abab, & ac \to acac, & ad \to acadab,\end{array}$$

and note that, for any word $w \in \{ab, ac, ad\}^*$ (representing an element of G_{i+1}) we have

$$\phi_i(\zeta_{\omega_i}(w)) \begin{cases}\langle\!\langle w, w\rangle\!\rangle & \text{if } \zeta_{\omega_i}(w) \text{ contains an even number of `}a\text{'}\\ \varepsilon\langle\!\langle wa, a^{-1}w\rangle\!\rangle & \text{if } \zeta_{\omega_i}(w) \text{ contains an odd number of `}a\text{'}.\end{cases}$$

In particular, ζ_x defines a homomorphism $G_{i+1} \to G_i$.

Proof of the lower bound. By induction, we see that for any non-trivial $w \in \{ab, ac, ad\}^*$ (representing an element of G_k) we have

$$\Delta_0(\zeta_{\omega_0}\cdots\zeta_{\omega_{k-1}}(w)) \geqslant 2^k.$$

Note then that, if $Z \in \mathbb{N}^3$ count the numbers of ab, ac, ad respectively in w, then $Z^{\mathrm{t}}M_x$ counts the numbers of ab, ac, ad respectively in $\zeta_x(w)$. Let $as \in \{ab, ac, ad\}$

be such that $\|s\|_k$ is minimal — if $\beta \leqslant \gamma, \delta$ then $s = b$, etc. Let W be the vector in $\mathbb{R}^3$ with a 1 at the position which s has in $\{ab, ac, ad\}$ and 0 elsewhere — if $\beta \leqslant \gamma, \delta$ then $W = (1,0,0)^{\mathrm{t}}$, etc. Set $w = \zeta_{\omega_0} \cdots \zeta_{\omega_{k-1}}(s)$. We have $\Delta_0(w) \geqslant 2^k$, and

$$\begin{aligned}\|w\|_0 &= W^{\mathrm{t}} M_{\omega_{k-1}} \cdots M_{\omega_0} V_0 \\ &= \eta_0 \cdots \eta_{k-1} W^{\mathrm{t}} V_k = \eta_0 \cdots \eta_{k-1} \mu_k. \qquad \square\end{aligned}$$

3.3. **Choice of inverted orbits growth.** If w be a word of norm R over G, then its inverted orbit $\mathcal{O}_w$ is a subset of X of cardinality at most $R+1$, and containing $*$. Furthermore, if w is a word over S, then $\mathcal{O}_w$ is contained in the ball of radius R in the Schreier graph of $(X, *)$. The set $\{\mathcal{O}_w \mid w \in S^R\}$ is therefore a finite subset of the power set of X, and we denote its cardinality by $\Sigma(R)$.

Proposition 3.10. *There is an absolute constant D such that for all $k \in \mathbb{N}$ we have*

$$\Sigma(\eta_0 \cdots \eta_{k-1} \mu_k) \leqslant D^{2^k}.$$

Proof. Again we denote by $\Sigma_i(R)$ the choice of inverted orbits growth function of G_i. The relevant inequality is

$$\Sigma_i(\eta_i R) \leqslant \int_{\ell + m = R + \|a\|_i / \eta_i} \Sigma_{i+1}(\ell) \Sigma_{i+1}(m),$$

and the same argument as in Proposition 3.6 applies. $\square$

4. Growth of the groups W_ω

Recall that W_ω is the permutational wreath product of a group A with G_ω. We estimate the growth of W_ω in terms of its inverted orbit growth as follows:

Lemma 4.1. *Let A have growth v_A, and let G have growth v_G. Let the inverted orbit growth of G on $(X, *)$ be Δ, and let its inverted orbit choice growth be Σ. Assume that v_A is log-concave. Consider $W = A \wr_X G$. Then*

$$v_G(R) v_A(R/\Delta(R))^{\Delta(R)} \precsim v_W(R) \precsim v_G(R) v_A(R/\Delta(R))^{\Delta(R)} \Sigma(R).$$

Proof. We begin by the lower bound. For $n \in \mathbb{N}$, consider a word w of norm R realizing the maximum $\Delta(R)$; write $\mathcal{O}(w) = \{x_1, \dots, x_k\}$ for $k = \Delta(R)$. Choose then k elements $a_1, \dots, a_k$ of norm $\leqslant R/k$ in A. Define $f \in \sum_X A$ by $f(x_i) = a_i$, all unspecified values being 1. Then $wf \in W$ may be expressed as a word of norm $R + |a_1| + \cdots + |a_k| \leqslant 2R$ in the standard generators of W.

Furthermore, different choices of a_i yield different elements of W; and there are $v_A(R/k)^k$ choices for all the elements of A. This proves the lower bound.

For the upper bound, consider a word w of norm R in W, and let $f \in \sum_X A$ denote its value in the base of the wreath product. The support of f has cardinality at most $\Delta(R)$, and may take at most $\Sigma(R)$ values.

Write then $\sup(f) = \{x_1, \dots, x_k\}$ for some $k \leqslant \Delta(R)$, and let $a_1, \dots, a_k \in A$ be the values of w at its support; write $\ell_i = \|a_i\|$. Since $\sum \ell_i \leqslant R$, the norms of the different elements on the support of f define a composition of a number not greater than R into at most k summands; such a composition is determined by k "marked positions" among $R + k$, so there are at most $\binom{R+k-1}{k}$ possibilities, which we bound crudely by $R^k/k!$. Furthermore, if A is finite, then no such composition occurs in the count, because the norms of the a_i are bounded. Each of the a_i is

then chosen among $v_A(\ell_i)$. elements, and again (by the assumption that v_A is log-concave) there are $\prod v_A(\ell_i) \leqslant v_A(R/k)^k$ total choices for the elements in A. In all cases, therefore, the $\binom{R+k-1}{k}$ term is absorbed by $v_A(R/k)^k$: if A is finite, then as we argued there is no binomial term, while if A is infinite then $v_A(R) \gtrsim R$.

We have now decomposed w into data that specify it uniquely, and we multiply the different possibilities for each of the pieces of data. Counting the possibilities for the value of w in G, the possibilities for its support in X, and the possibilities for the elements in A, we get

$$v_W(R) \precsim v_G(R)v_A(R/k)^k\Sigma(R),$$

which is maximized by $k = \Delta(R)$. □

The previous section then shows:

Corollary 4.2. *There are two absolute constants $E > 1, F$ such that the growth function v of $W_\omega = A \wr_X G_\omega$ satisfies*

$$E^{2^k} \leqslant v(\eta_0 \cdots \eta_{k-1}\mu_k) \leqslant F^{2^k}.$$

Proof. Take together the upper bound on the growth of G_ω from Proposition 3.6, the bounds on the inverted orbit growth from Proposition 3.9, and the choices for the inverted orbits from Proposition 3.10. The conclusion follows from Lemma 4.1. □

We now estimate what the growth of W_ω for periodic sequences ω. We start by the easy

Lemma 4.3. *Let $v, v' : \mathbb{N} \to \mathbb{N}$ be increasing functions such that $v(R_t) \leqslant v'(R_t)$ for an strictly increasing sequence $R_1, R_2, \dots$ with R_{t+1}/R_t bounded. Then $v \precsim v'$.*

Proof. Say $R_{t+1}/R_t \leqslant K$ for all $t \in \mathbb{N}$. Given $R \in \mathbb{N}$, let $t \in \mathbb{N}$ be such that $R_{t-1} < R \leqslant R_t$. Then

$$v(R) \leqslant v(R_t) \leqslant v'(R_t) \leqslant v'(KR),$$

so $v \precsim v'$. □

The following result isolates a special case of Theorem A; we include it because it constructs groups with additional properties (recursive presentation, self-similarity).

Proposition 4.4. *Let $\omega = \overline{\omega_0 \dots \omega_{k-1}}$ be a periodic sequence. Let $\eta \in (2, 3)$ be such that*

$$\eta^k = \operatorname{sp.radius}(M_{\omega_{k-1}} \cdots M_{\omega_0}).$$

Then the group W_ω has growth

$$v_\omega(R) \sim \exp(R^{\log 2/\log \eta}).$$

Proof. We choose $V_0 \in \Delta$ to be an eigenvector for $M_{\omega_{k-1}} \cdots M_{\omega_0}$. Then its eigenvalue is $\eta_{k-1} \cdots \eta_0$, and coincides with its spectral radius. Indeed the maps $\overline{M}_x$ are contracting on Δ, so $M_{\omega_{k-1}} \cdots M_{\omega_0}$ has precisely one real eigenvalue. We get $\eta_{k-1} \cdots \eta_0 = \eta^k$. It suffices to estimate the growth function $w(R)$ of W_ω at exponentially-spaced values $(\eta_{k-1} \cdots \eta_0)^t$ for $t \in \mathbb{N}$, by Lemma 4.3, and at these places we have, by Corollary 4.2,

$$E^{2^{kt}} \leqslant v(\eta^{kt}\mu_0) \leqslant F^{2^{kt}} \quad \text{for all } t \in \mathbb{N},$$

so, for all $R_t = \eta^{kt}\mu_0$, we get

$$E^{(R_t/\mu_0)^{\log 2/\log\eta}} \leqslant v(R_t) \leqslant F^{(R_t/\mu_0)^{\log 2/\log\eta}}. \qquad \square$$

Note that, because (V_0,ω) is periodic, the V_i define a discrete sequence in Δ and in particular do not accumulate on its boundary, so the function μ is bounded from below on $\{V_i\}$. More care is needed in the general case.

4.1. **Dynamics on the simplex.** We now show that the spectral radii in Proposition 4.4 are dense in the interval $[2,\eta_+]$. For that purpose, it is useful to translate the problem to a slightly different language.

Let f be the projection of s^{-1} to Δ'. This is a 3-to-1 map, and is expanding for the Hilbert metric on Δ, because the $\overline{M}_x$ are contracting. Periodic orbits under f correspond bijectively to s-periodic orbits in Ω^+, by reading them backwards. Indeed, the ω-coördinate can be uniquely recovered by noting in which subsimplex of Δ' the point lies.

More precisely, if $f^kq = q$, then for $i \in \{0,\dots,k\}$ let ω_i be such that $f^{k-i}(q)$ belongs to the image of $\overline{M}_{\omega_i}$, and extend the sequence ω periodically. Then the f-orbit of q is the reverse of the s-orbit of (q,ω).

Recall that η is a continuous function on the simplex; it equals 2 on the boundary, and 3 at the barycentre of Δ. Write $\theta(p) = \log\eta(p)$. For a periodic point p, of period k, write $\theta^+(p)$ the Cesarò average of θ on p:

$$\theta^+(p) := \frac{1}{k}\big(\theta(p) + \theta(fp) + \cdots + \theta(f^{k-1}p)\big).$$

Lemma 4.5. *Let p be an f-periodic point of period k in Δ'. Let $\omega_0,\dots,\omega_{k-1}$ be the corresponding address. Then the spectral radius of $M_{\omega_{k-1}}\cdots M_{\omega_0}$ equals $k\theta^+(p)$.*

Proposition 4.6. *The averages $\theta^+(p)$ are dense in $[\log 2, \log\eta_+]$.*

Proof. Consider $p,p' \in \Sigma$ two periodic points, say of period k,k' respectively.

Because f is expanding, there exist arbitrarily small open sets $\mathcal{U} \subset \Delta$ containing p, such that $\mathcal{U} \subset f^k(\mathcal{U})$ and f^k uniformly expands on $\mathcal{U}$; similarly $f^{k'}$ uniformly expands on the neighbourhood $\mathcal{U}' \subset \Delta$ of p'. Without loss of generality, we assume $\mathcal{U}$ and $\mathcal{U}'$ are relatively compact.

For any finite sequence $\omega_1\dots\omega_n \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}^n$, consider the fixed point p_ω of $\overline{M}_{\omega_n}\cdots\overline{M}_{\omega_1}$. The triangles $\overline{M}_{\omega_n}\cdots\overline{M}_{\omega_1}(\Delta)$ become arbitrarily small, as $n\to\infty$ and all three symbols occur in $\omega_1\dots\omega_n$; so periodic points are dense in Δ.

It then follows that, for every non-empty open set $\mathcal{O}$, we have $\bigcup_{n\geqslant 0} f^n(\mathcal{O}) = \Delta$; indeed $\bigcup_{n\geqslant 0} f^n(\mathcal{O})$ is open, and contains all periodic points because f is expanding.

Because θ is continuous on Δ, for any $\epsilon > 0$ we can make $\mathcal{U}$ small enough so that $\frac{1}{k}(\theta(q) + \theta(fq) + \cdots + \theta(f^{k-1}q))$ is less that ϵ away from $\theta^+(p)$, for all $q \in \mathcal{U}$. Similarly, $\mathcal{U}'$ may be chosen small enough that the average of θ on the first k' points of the orbit of q' is at most ϵ away from $\theta^+(p')$, for all $q' \in \mathcal{U}'$.

Since $\mathcal{U}'$ is relatively compact, there exists $\ell \in \mathbb{N}$ such that $\mathcal{U}' \subset f^\ell(\mathcal{U})$ and f^ℓ uniformly expands on $\mathcal{U}$; similarly $\mathcal{U} \subset f^{\ell'}(\mathcal{U}')$ for some $\ell' \in \mathbb{N}$.

Consider now $n,n' \in \mathbb{N}$. We have locally defined contractions $f^{-k}\colon \mathcal{U} \to \mathcal{U}$, $f^{-\ell}\colon \mathcal{U}' \to \mathcal{U}$, $f^{-k'}\colon \mathcal{U}' \to \mathcal{U}'$ and $f^{-\ell'}\colon \mathcal{U} \to \mathcal{U}'$, which we compose:

$$\mathcal{U} \xrightarrow{f^{-\ell'}} \mathcal{U}' \xrightarrow{f^{-n'k'}} \mathcal{U}' \xrightarrow{f^{-\ell}} \mathcal{U} \xrightarrow{f^{-nk}} \mathcal{U}.$$

This is a contraction $\mathcal{U} \to \mathcal{U}$, so by the Banach fixed point theorem there exists a fixed point $q \in \mathcal{U}$ for the composite $f^{\ell+nk+\ell'+n'k'}$ that remains close to the orbit of p for nk steps, wanders for ℓ steps, remains close to the orbit of p' for $n'k'$ steps, and wanders back to q for ℓ' steps.

The average of θ on q is

$$\theta^+(q) = \frac{1}{nk+\ell+n'k'+\ell'}\big(\theta(q)+\cdots+\theta(f^{nk+\ell+n'k'+\ell'-1}q)\big).$$

The orbit of q is, except at $\ell+\ell'$ instants, either close to the orbit of p or close to the orbit of p'. Consider any $\rho \in \mathbb{R}_+$. Then, as $n, n' \to \infty$ with ratio $n/n' \to \rho$, we have

$$\liminf_{n,n'\to\infty} \theta^+(q) \leqslant \frac{\rho k(\theta^+(p)+\epsilon)+k'(\theta^+(p')+\epsilon)}{\rho k+k'},$$

and

$$\limsup_{n,n'\to\infty} \theta^+(q) \geqslant \frac{\rho k(\theta^+(p)-\epsilon)+k'(\theta^+(p')-\epsilon)}{\rho k+k'}.$$

Letting ϵ tend to 0 and simultaneously considering all possible ρ, we obtain a dense set of values in $[\theta^+(p), \theta^+(p')]$. More precisely, for any $t \in [\theta^+(p), \theta^+(p')]$, let ρ be such that $(\rho k\theta^+(p)+k'\theta^+(p'))/(\rho k+k') = t$; then for any $\epsilon > 0$ there exists $\mathcal{U}, \mathcal{U}'$ as above; then ℓ, ℓ' as above; and finally n, n' large enough so that $|\theta^+(q)-t| < 2\epsilon$.

Finally, note that we can take for p the periodic orbit corresponding to the sequence $\omega = \overline{\mathbf{012}}$; while for p' we consider the periodic point corresponding to the sequence $\omega' = \overline{\mathbf{0}^u\mathbf{12}}$ for u sufficiently large; we have $\theta^+(p') \to \log 2$ as $u \to \infty$. $\square$

4.2. **Proof of Theorem B.** Let ω be a periodic sequence, and let A be a finite group. We already showed in Proposition 2.4 that W_ω is self-similar and branched. To show that it is contracting, we endow it with the following metric. It is generated by $\{1, a, b, c, d\} \times A$; for $s \in \{a, b, c, d\}$ and $t \in A$, the norm of st is $\|s\|$, while for $t \neq 1$ in A its norm is $\|a\|$.

We start by a more general result, which holds for arbitrary sequences ω. Recall from §3 that we set $G_i = G_{s^i\omega}$ and selected metrics $\|\cdot\|_i$ on G_i, giving constants η_i. We set $W_i = A \wr_X G_i$, with the metric above.

The reason we can achieve lower bounds on the growth of W_i may be illustrated as follows; though we will not use it directly. Consider the corresponding permutational wreath products $W_i = A \wr_X G_i$, and note that we also have injective homomorphisms $\psi_i \colon W_i \to W_{i+1} \wr \mathfrak{S}_2$. The maps ψ_i have the same Lipschitz property as ϕ_i, see Lemma 4.7. Additionally, for all elements of W_i with sufficiently large support (and there are sufficiently many so as to dominate the asymptotics), we have a reverse inequality $\|\psi_i(g)\| \geqslant 2/\eta_i\|g\| - C$.

The following combines Lemma 3.3 and [6, Lemma 4.2]; we only sketch the proof since it follows closely that of its models.

Lemma 4.7. *For each $i \geqslant 0$ and each $g \in W_i$, with $\psi_i(g) = \langle\langle g_1, g_2\rangle\rangle\pi$, we have*

$$\|g_0\|_{i+1} + \|g_1\|_{i+1} \leqslant 2\|a\|_{i+1} + \frac{2}{\eta_i}(\|g\|_i + \|a\|_i).$$

Proof. Consider a minimal representation $g = s_1t_1\dots s_nt_n$ of $g \in W_i$ in the generating set $\{1, a, b, c, d\} \times A$. Recall that we have $\psi_i : W_i \to W_{i+1} \wr \mathfrak{S}_2$, defined on generators by $\psi_i(s) = \phi_i(s)$ for $s \in \{a, b, c, d\}$, and $\psi_i(t) = \langle\langle 1, t\rangle\rangle$ for $t \in A$. Therefore, $\langle b, c, d\rangle$ commutes with A in W_i; so (by minimality) we may assume that no two consecutive s_j, s_{j+1} belong to $\{b, c, d\}$.

It follows that we have $g = a^{\epsilon}x_1t_1ax_2t_2\dots ax_mt_m$, with $\epsilon \in \{0,1\}$, all $x_2,\dots,x_{m-1} \in \{b,c,d\}$, $x_1, x_m \in \{1,b,c,d\}$, and $m \leqslant (n+1)/2$. We then proceed as in Lemma 3.3, to construct words representing g_1, g_2. Each t_j contributes a t_j to either g_1 or g_2; each x_j contributes a letter in $\{1,a,b,c,d\}$ to each of g_1 and g_2. Therefore, g_1 has the form $u_0y_1u_1\dots y_mu_m$ for some $u_j \in A$ and $y_i \in \{1,a,b,c,d\}$, while g_2 has the similar form $v_0z_1v_1\dots z_mv_m$. Furthermore, $\|y_j\|_{i+1} + \|z_j\|_{i+1} \leqslant 2/\eta_i\|ax_j\|$, by (3). Then $\|g_1\|_{i+1} + \|g_2\|_{i+1} \leqslant \|u_0\|_{i+1} + \|v_0\|_{i+1} + 2/\eta_i(\|g\|_i + \|a\|_i)$, as was to be shown. □

We finally recall a classical estimate of growth as a function of sum-contraction:

Lemma 4.8 (See e.g. [1, Proposition 4.3]). *If G is self-similar and sum-contracting with contraction η, then $v_G \precsim \exp(R^{\log d/\log\eta})$.*

Proof of Theorem B. Lemma 4.7 gives an upper bound on the growth of W_ω; the same upper bound comes from Proposition 4.4.

The density of the growth exponents in $[\alpha_-, 1]$ follows from Proposition 4.6.

To obtain torsion-free examples, apply Lemma 4.1, recalling that the growth of $\tilde{G}_\omega$ is equivalent to that of $\mathbb{Z} \wr G_\omega$, see §2.3. □

5. Proof of Theorem A

We now obtain more growth functions by considering W_ω and $\tilde{G}_\omega$ for non-periodic sequences ω. We first give an estimate for μ_k; only the first part of Lemma 5.1 will be used.

For a finite sequence $\omega = \omega_0\dots\omega_{n-1} \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}^n$ and $p \in \Delta$, we write by extension

$$\overline{M}_\omega = \overline{M}_{\omega_0}\cdots\overline{M}_{\omega_{n-1}}\colon \Delta \to \Delta$$

and

$$\eta(p,\omega_0\dots\omega_{n-1}) = \eta(p,\omega_0)\eta(\overline{M}_{\omega_0}p,\omega_1)\cdots\eta(\overline{M}_{\omega_1\dots\omega_{n-2}}p,\omega_{n-1}).$$

Fix also once and for all $V_0 \in \Delta'$. When a sequence ω is under consideration, it defines $V_k \in \Delta'$ and η_k, μ_k by $V_{k+1} = \overline{M}_{\omega_k}(V_k)$ and $\eta_{k+1} = \eta(V_k,\omega_k)$ and $\mu_k = \mu(V_k)$.

Lemma 5.1. *Consider $\omega \in \{\mathbf{0},\mathbf{1},\mathbf{2}\}^\infty$. For $k \in \mathbb{N}$, let $\ell \in \mathbb{N}$ be maximal such that $\omega_{k-\ell+1}\dots\omega_k$ contains only two of the symbols $\mathbf{0},\mathbf{1},\mathbf{2}$, say $\mathbf{a},\mathbf{b}$. Furthermore write $\omega_{k-\ell+1}\dots\omega_k = \mathbf{a}^{i_1}\mathbf{b}^{j_1}\dots\mathbf{a}^{i_m}\mathbf{b}^{j_m}$ with m minimal. Then*

$$K/m \leqslant \mu_k \leqslant L/m$$

for absolute constants K, L.

Proof. Without loss of generality, we assume $\omega = \dots\mathbf{0}\mathbf{1}^t\mathbf{2}\mathbf{0}^{i_1}\dots\mathbf{1}^{j_m}$, with $i_1+\dots+j_m = \ell$. We have $M_{\mathbf{0}^n} = \begin{pmatrix}1 & 2^n-1 & 2^n-1\\ 0 & 2^n & 0\\ 0 & 0 & 2^n\end{pmatrix}$.

The image of $\overline{M}_{\mathbf{0}}$ is the open triangle spanned by $(\frac13,\frac13,\frac13)$, $(\frac12,0,\frac12)$ and $(\frac12,\frac12,0)$. The image of $\overline{M}_{\mathbf{0}\mathbf{1}^t}$ is contained in the open triangle spanned by $(\frac14,\frac12,\frac14)$, $(\frac13,\frac13,\frac13)$ and $(\frac12,\frac12,0)$. The image of $\overline{M}_{\mathbf{0}\mathbf{1}^t\mathbf{2}}$ is contained in the open triangle spanned by $(\frac15,\frac25,\frac25)$, $(\frac27,\frac27,\frac27)$ and $(\frac13,\frac13,\frac13)$. It follows that $V_{k-\ell}$ belongs to that triangle, so $\mu_{k-\ell} \in (\frac15,\frac13)$ is bounded away from 0.

Now, under application of $\mathbf{0},\mathbf{1}$, the third coördinate of $V_{k-\ell+n}$ decreases as n increases, while the first two approach $1/2$. To understand how exactly, we consider $V=(1/2-\rho\epsilon, 1/2-(1-\rho)\epsilon,\epsilon)$ for some $\epsilon\ll 1$ and $\rho\in(\frac12,1)$, and compute

$$\overline{M}_{\mathbf{0}^n}(V)=\begin{pmatrix}\frac12-2^{-n}\rho\epsilon+\mathcal{O}(\epsilon^2)\\ \frac12-(1-2^{-n}\rho)\epsilon+\mathcal{O}(\epsilon^2)\\ \epsilon-2\rho(1-2^{-n})\epsilon^2+\mathcal{O}(\epsilon^3)\end{pmatrix}. \tag{6}$$

A similar approximation holds for $\overline{M}_{\mathbf{1}^n}(V)$, with the first two rows switched.

Write $W=\overline{M}_{\mathbf{0}^n}(V)$. It follows that $\mu(W)\approx\epsilon-A\epsilon^2$ is bounded away from 0 as $n\to\infty$, with $A\in(\frac12,1)$. On the other hand, if $n\geqslant 1$, then W is of the form $(1/2-(1-\rho')\epsilon', 1/2-\rho'\epsilon',\epsilon')$ for $\epsilon'\approx\epsilon-A\epsilon^2$ and some $\rho'\in(\frac12,1)$. Set $X=\overline{M}_{\mathbf{1}^p}(W)$; then $\mu(X)\approx\epsilon'-2\rho'(1-2^{-p})(\epsilon')^2=\epsilon'-B(\epsilon')^2$ for some $B\in(\frac12,1)$.

If we now translate to coördinates $\epsilon=1/N$, we get $\mu(V)=1/N$, $\mu(W)\approx 1/(N+A)$ and $\mu(X)\approx 1/(N+A+B)$ with $A,B\in(\frac12,1)$. We repeat this m times, giving $\mu(V_k)\approx 1/(N+A_1+B_1+\cdots+A_m+B_m)$ with $A_i,B_i\in(\frac12,1)$ if $\mu(V_{k-\ell})\approx 1/N$. This translates to $K=1,L=2$ in the statement of the lemma. In fact, the constants are a bit worse because of the approximations we made in (6), that are accurate only for small μ. □

Corollary 5.2. *If the sequence ω has the form*

$$\omega=(\mathbf{012})^{i_1}\mathbf{2}^{j_1}(\mathbf{012})^{i_2}\mathbf{2}^{j_2}(\mathbf{012})^{i_3}\mathbf{2}^{j_3}\dots, \tag{7}$$

with $i_1,j_1,i_2,j_2,\dots\geqslant 1$, then the μ_k are all bounded away from 0.

Proof. In fact, the image of $\overline{M}_{\mathbf{012}}$ is the open triangle spanned by $(\frac13,\frac13,\frac13)$, $(\frac27,\frac27,\frac37)$ and $(\frac{4}{17},\frac{6}{17},\frac{7}{17})$, so after each $\mathbf{012}$ the μ_k belongs to $(\frac{4}{17},\frac13)$.

The image of that triangle under $\overline{M}_{\mathbf{2}^i}$ is contained in the convex quadrilateral spanned by $(\frac13,\frac13,\frac13)$, $(\frac{4}{17},\frac{6}{17},\frac{7}{17})$, $(\frac14,\frac14,\frac12)$ and $(\frac15,\frac{3}{10},\frac12)$, so $\mu_k\in(\frac15,\frac13)$ for all k. □

We also show that η converges very fast to its limiting values under periodic orbits:

Lemma 5.3. *There exist constants $A'\leqslant 1$, $B'\geqslant 1$ such that*

(1) *For all $V\in\Delta'$ and all $n\in\mathbb{N}$,*
$$\eta(V,(\mathbf{012})^n)\geqslant\eta_+^{3n}A';$$
(2) *For all $V\in\Delta'$ and all $n\in\mathbb{N}$,*
$$\eta(V,\mathbf{2}^n)\leqslant 2^nB'.$$

Proof. Let $V_+\in\Delta$ denote the fixed point of $\overline{M}_{\mathbf{012}}$. Note first that $\eta(-,\mathbf{012})$ is differentiable at V_+, and that $\overline{M}_{\mathbf{012}}$ is uniformly contracting about V_+. Let $\mathcal{U}$ be a neighbourhood of V_+ such that $\overline{M}_{\mathbf{012}}$ is ρ-Lipschitz in $\mathcal{U}$, and let D be an upper bound for the derivative of $\log\eta(-,\mathbf{012})$ on $\mathcal{U}$. We may assume, without loss of generality, that $\mathcal{U}$ contains the image of $\overline{M}_{\mathbf{012}}$. Recall that $\eta_+^3=\eta(V_+,\mathbf{012})$.

For all $k\in\mathbb{N}$, write $V_k=\overline{M}_{(\mathbf{012})^k}(V)$. For $k\geqslant 1$ we have $d(V_k,V_+)\leqslant\rho^{k-1}$, so $|\log\eta(V_k,\mathbf{012})-3\log\eta_+|<D\rho^{k-1}$, while for $k=0$ we write $|\log\eta(V,\mathbf{012})-3\log\eta_+|<3\log 3$. Therefore,

$$|\log\eta(V,(\mathbf{012})^n)-3n\log\eta_+|\leqslant 3\log 3+\sum_{k=1}^{n-1}|\log\eta(V_k,\mathbf{012})-3\log\eta_+|\leqslant 3\log 3+D/(1-\rho)$$

is bounded over all n and V. The estimate (1) follows, with $A' = 3^3 \exp(D/(1-\rho))$.

For the second part, consider $V_k = \overline{M}_{\mathbf{2}^k}(V)$, and note that V_k converges to a point V_∞ on the side $\{\delta = \frac{1}{2}\}$ of Δ. By the approximations (6), we get $d(V_k, V_\infty) \approx 2^{-k}$, so definitely $d(V_k, V_\infty) < \rho^{k-1}$ for some $\rho < 1$. As above, $\eta(-, \mathbf{2})$ is differentiable in a neighbourhood $\mathcal{U}$ of $\{\delta = \frac{1}{2}\}$, and the derivative of $\log \eta(-, \mathbf{2})$ is bounded on $\mathcal{U}$, say by D. We may again assume, without loss of generality, that $\mathcal{U}$ contains the image of $\overline{M}_{\mathbf{2}}$. Recall that $\eta(V, \mathbf{2}) = 2$ for all V on $\{\delta = \frac{1}{2}\}$.

As before, for $k \geqslant 1$ we have $|\log \eta(V_k, \mathbf{2}) - \log 2| < D\rho^{k-1}$, while for $k = 0$ we write $|\log \eta(V, \mathbf{2}) - \log 2| < \log 3$. Therefore,

$$|\log \eta(V, \mathbf{2}^n) - n \log 2| \leqslant \log 3 + \sum_{k=1}^{n-1} |\log \eta(V_k, \mathbf{2}) - \log \eta_+| \leqslant \log 3 + D/(1-\rho)$$

is bounded over all n and V. The estimate (1) follows, with $B' = 3 \exp(D/(1-\rho))$. □

We now reformulate the statement of Theorem A as follows. For the main statement, set $g(R) = \log f(R)$; to construct a torsion-free group as in Remark 1.1, define g implicitly by $(R/g(R))^{g(R)} = f(R)$. In both cases, we obtain

$$g(2R) \leqslant 2g(R) \leqslant g(\eta_+ R) \tag{8}$$

for all R large enough. For simplicity (since growth is only an asymptotic property) we assume that (8) holds for all R. Without loss of generality, we also assume that g is increasing and satisfies $g(1) = 1$.

Recall that an initial metric $V_0 \in \Delta'$ has been chosen. We will construct a sequence ω such that, for constants A, B, we have

$$A \leqslant \frac{g(\eta(V_0, \omega_0 \dots \omega_{k-1}))}{2^k} \leqslant B \text{ for all } k; \tag{9}$$

in fact, it will suffice to obtain this inequality for a set of values $k_0, k_1, \dots$ of k such that $\sup_i(k_{i+1} - k_i) < \infty$. Indeed, the orbit V_i of V_0 in Δ' will remain bounded, so we will have $\mu(V_i) \in [C, 1]$ for some $C > 0$. Consider for simplicity the first part of the theorem, for which $g(R) = \log f(R)$. Then, by Corollary 4.2,

$$E^{2^k} \leqslant v(\eta(V_0, \omega_0 \dots \omega_{k-1})\mu_k) \leqslant v(Bg^{-1}(2^k)),$$
$$v(ACg^{-1}(2^k)) \leqslant v(\eta(V_0, \omega_0 \dots \omega_{k-1})\mu_k) \leqslant F^{2^k}$$

and therefore $v(R) \sim \exp(g(R)) = f(R)$. Similar considerations hold for the torsion-free case of Remark 1.1, using Lemma 4.1.

Proof of (9). We will construct a sequence ω of the form (7),

$$\omega = (\mathbf{012})^{i_1} \mathbf{2}^{j_1} (\mathbf{012})^{i_2} \mathbf{2}^{j_2} (\mathbf{012})^{i_3} \mathbf{2}^{j_3} \dots,$$

with $i_1, j_1, i_2, j_2, \dots \geqslant 1$. The μ_k are bounded away from 0 by Corollary 5.2.

We start by the empty sequence. Then, assuming $\omega' = \omega_0 \dots \omega_{k-1}$ has been constructed, we repeat the following:

- while $g(\eta(V_0, \omega')) < 2^k$, we append $\mathbf{012}$ to ω';
- while $g(\eta(V_0, \omega')) > 2^k$, we append $\mathbf{2}$ to ω'.

For our induction hypothesis, we assume that the stronger condition

$$\frac{1}{2} 2^k \leqslant g(\eta(V_0, \omega')) \leqslant 2^k$$

holds for each k of the form $i_1 + j_1 + \cdots + i_m + j_m$, and that

$$2^k \leqslant g(\eta(V_0, \omega')) \leqslant 3^3 2^k$$

holds for each k of the form $i_1 + j_1 + \cdots + i_m$; these conditions apply whenever ω is a product of 'syllables' $(\mathbf{012})^{i_t}$ and $\mathbf{2}^{j_t}$.

Consider first the case $\frac{1}{2}2^k \leqslant g(\eta(V_0, \omega')) \leqslant 2^k$; and let n be minimal such that $g(\eta(V_0, \omega'(\mathbf{012})^n)) > 2^{k+3n}$. Then, for all $i \in \{1, \dots, n\}$, Lemma 5.3(1) gives $\eta(V_0, \omega'(\mathbf{012})^i) \geqslant \eta(V_0, \omega')\eta_+^{3i} A'$. Let $u \in \mathbb{N}$ be minimal such that $A' \geqslant \eta_+^{-u}$; this, combined with $g(\eta_+ R) \geqslant 2g(R)$, gives

$$g(\eta(V_0, \omega'(\mathbf{012})^i)) \geqslant g(\eta(V_0, \omega')\eta_+^{3i-u}) \geqslant 2^{-1-u} 2^{k+3i}.$$

By minimality of n, we have $g(\eta(V_0, \omega'(\mathbf{012})^{n-1})) \leqslant 2^{k+3(n-1)}$; since g is sublinear and $\eta \leqslant 3$, we get

$$g(\eta(V_0, \omega'(\mathbf{012})^n)) \leqslant 3^3 2^{k+3n}.$$

Consider then the case $2^k \leqslant g(\eta(V_0, \omega')) \leqslant 3^3 2^k$, which is completely symmetric, and let n be minimal such that $g(\eta(V_0, \omega'\mathbf{2}^n)) < 2^{k+n}$. Then, for all $i \in \{1, \dots, n\}$, Lemma 5.3(2) gives $\eta(V_0, \omega'\mathbf{2}^i) \leqslant \eta(V_0, \omega')2^i B'$; this, combined with $g(2R) \leqslant 2g(R)$, gives

$$g(\eta(V_0, \omega'\mathbf{2}^i)) \leqslant 3^3 2^{k+i} B'.$$

By minimality of n, we have $g(\eta(V_0, \omega'\mathbf{2}^{n-1})) \geqslant 2^{k+n-1}$; since g is increasing, we get

$$g(\eta(V_0, \omega'\mathbf{2}^n)) \geqslant \frac{1}{2} 2^{k+n}.$$

We have proved the claim (9), with $A = 2^{-1-u}$ and $B = 3^3 B'$. □

Remark 5.4. The construction of ω from f is algorithmic, in the following sense. If V_0 is chosen with rational coëfficients, then all V_k have rational coëfficients, and therefore are be computable. Furthermore, $\eta(V_0, \omega_0 \dots \omega_{k-1})$ is also computable. Therefore, if f is recursive, then so is ω.

It then follows that G_ω, $\tilde{G}_\omega$ and $W_\omega = A \wr_X G_\omega$ are recursively presented (for recursively presented A), see §2.2 and §2.4.

5.1. **Illustrations.** Given a sufficiently regular growth function f, Theorem A constructs a sequence ω such that the growth of W_ω is asymptotically f. We may proceed the other way round, and consider 'regular' sequences ω, using then Theorem A in reverse to estimate coarsely the growth of W_ω. Here are four examples:

- Consider the sequence $\omega = (\mathbf{012})\mathbf{2}^1(\mathbf{012})\mathbf{2}^2(\mathbf{012})\mathbf{2}^3(\mathbf{012})\mathbf{2}^4 \dots$. Among the first k entries, approximately $\sqrt{k}$ instances of $\mathbf{012}$ will have been seen; therefore $\eta(V_0, \omega_0 \dots \omega_{k-1}) \approx 2^{k+\mathcal{O}(1)\sqrt{k}}$. This gives a growth function of the rough order of
$$\exp(R/\exp\sqrt{\log R}).$$
- Consider the sequence $\omega = (\mathbf{012})\mathbf{2}^1(\mathbf{012})\mathbf{2}^2(\mathbf{012})\mathbf{2}^4(\mathbf{012})\mathbf{2}^8 \dots$. Among the first k entries, approximately $\log k$ instances of $\mathbf{012}$ will have been seen; therefore $\eta(V_0, \omega_0 \dots \omega_{k-1}) \approx 2^{k+\mathcal{O}(1)\log k}$. This gives a growth function of the rough order of
$$\exp(R/\log R).$$

- Consider the sequence $\omega = (\mathbf{012})\mathbf{2}^{2^1}(\mathbf{012})\mathbf{2}^{2^2}(\mathbf{012})\mathbf{2}^{2^4}(\mathbf{012})\mathbf{2}^{2^8}\dots$. Among the first k entries, approximately $\log\log k$ instances of $\mathbf{012}$ will have been seen; therefore $\eta(V_0, \omega_0 \dots \omega_{k-1}) \approx 2^{k+\mathcal{O}(1)\log\log k}$. This gives a growth function of the rough order of
$$\exp(R/\log\log R).$$
- Consider the Ackermann function
$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0,\\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0,\\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0,\end{cases}$$
and consider $\omega = (\mathbf{012})\mathbf{2}^{A(0,0)}(\mathbf{012})\mathbf{2}^{A(1,1)}(\mathbf{012})\mathbf{2}^{A(2,2)}\dots$. Then W_ω is a group of subexponential growth, whose growth is larger than any primitive recursive function of subexponential growth.

Thanks. We express our gratitude to Bill Thurston for explanations and discussions on the topic of dynamics on the simplex and on the density of η-averages as in Theorem B, which he generously contributed during a conference in Roskilde in September 2010.

We are also grateful to Slava Grigorchuk and Pierre de la Harpe for their comments on a previous version of this article, that helped improve its readability.

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